Int. J. Solids Structures Vol. 15, pp. 607–614
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0020-7683/79/0801-0607/$02.00/0

# SOLUTION OF TIME DEPENDENT BOUNDARY VALUE PROBLEMS BY THE BOUNDARY OPERATOR METHOD

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(*Received* 13 *September* 1978; *in revised form* 14 *December* 1978)

**Abstract**—A new procedure, the Boundary Operator Method (BOM), for deriving normal mode response formulas in systems with time dependent boundary conditions is demonstrated. The method of analysis is applied to a finite, nonhomogeneous bar. This example illustrates the following advantages of the BOM: (1) The resulting formulas for the displacement and stress are valid for the entire history of the system and for Dirichlet, Neumann or Cauchy boundary conditions. (2) The uniqueness of the solution is unobscured since, unlike the Mindlin–Goodman method, the present solution technique does not require specific spatial functions which render the transformed boundary conditions homogeneous. (3) The evaluation of the solution is expedited by the absence of integrals containing auxiliary spatial functions. (4) The analysis employs standard mathematical techniques.

## NOMENCLATURE

| | |
|---|---|
| $A_k, B_k, C_k$ | constants, $F^0L^0T^0$ |
| $c_0 = (E_0/\rho)$ | elastic wave velocity, $F^0L^1T^{-1}$ |
| $c_m$ | constant, $F^0L^0T^0$ |
| $D_i$ | $\cos\alpha_i + \sin\alpha_i(\partial/\partial\epsilon)$, $F^0L^0T^0$ |
| $f_i$ | dimensionless temporal function, $F^0L^0T^0$ |
| $g_i$ | dimensionless spatial function, $F^0L^0T^0$ |
| $I_k = \int_{\epsilon_1}^{\epsilon_2} X_k^2\, d\epsilon^*$ | $F^0L^0T^0$ |
| $L$ | length of bar, $F^0L^1T^0$ |
| $K_s$ | upper limit of summation in eqn (29), $F^0L^0T^0$ |
| $n$ | real exponent |
| $0_i$ | boundary operator defined in eqn (13), $F^0L^0T^0$ |
| $T_k$ | dimensionless temporal function, $F^0L^0T^0$ |
| $t$ | real time, $F^0L^1T^0$ |
| $u$ | displacement in $X$ direction, $F^0L^1T^0$ |
| $X_k$ | dimensionless eigenfunction, $F^0L^0T^0$ |
| $x$ | axial coordinate, $F^0L^1T^0$ |
| $\alpha_i$ | dimensionless real numbers, $F^0L^0T^0$ |
| $\beta_1 = \pi/\tau_0$ | $F^0L^0T^0$ |
| $\beta_2 = k\pi$ | $F^0L^0T^0$ |
| $\epsilon = X/\kappa L$ | dimensionless axial coordinate, $F^0L^0T^0$ |
| $\epsilon_1, \epsilon_2$ | dimensionless boundaries of bar, $F^0L^0T^0$ |
| $\zeta$ | dimensionless displacement, $F^0L^0T^0$ |
| $\kappa$ | dimensionless coefficient, $F^0L^0T^0$ |
| $\lambda_k$ | eigenvalues, $F^0L^0T^0$ |
| $\rho$ | mass density, $F^1L^{-4}T^2$ |
| $\sigma_0$ | dimensionless amplitude of stress pulse, $F^0L^0T^0$ |
| $\tau = c_0t/\kappa L$ | dimensionless time, $F^0L^0T^0$ |
| $\Psi = u/\kappa L$ | dimensionless displacement, $F^0L^0T^0$ |
| * | denotes dummy variable |
| / | denotes derivative *wrt* $\epsilon$ |
| · | denotes derivative *wrt* $\tau$ |

## INTRODUCTION

A variety of computational techniques have been employed to analyze one dimensional, finite, continuous systems subjected to time dependent boundary conditions. Cinelli[1] has ascertained the transient response of thick elastic cylinders and spheres to dynamic surface loadings by employing a finite Hankel transform. After a transformation that resulted in homogeneous boundary conditions, Lee[2] also invoked the finite Hankel transform to study wave propagation in a nonhomogeneous rod. Schreyer[3] has developed an inverse procedure that yields exact solutions to the one-dimensional inhomogeneous wave equation. Reference[3] employed these transformation relations to compute amplitude amplification vs dimensionless frequency at various locations in a bounded gradient layer resting on sinusoidally excited bedrock. The method of characteristics was applied to the class of problems under discussion in Chou and Greif[4] and Greif and Chou[5]. In Alzheimer and Forrestal[6], the circumferential stress history at the inner surface of several thick cylinders subjected to a step pressure was calculated using the Laplace transform technique in conjunction with asymptotic expansion of the transform variable. Good agreement for the early time response was demonstrated between these results and the numerical studies of [4]. Lindholm and Doshi[7] utilized the method of virtual work to quantify the response of an elastic, nonhomogeneous, finite bar to a pressure pulse. This procedure was also employed by Fisher[8] in a study of wave propagation in a hollow conical frustum. Recently, Pao and Ceranoglu[9] have studied thick-wall spherical shells using ray theory. Numerical results presented in [9] quantify the amplification and the compression-tension oscillation of the radial stress history at various points in the hollow sphere.

A general method that is applicable to all the problems discussed above is described in Mindlin and Goodman[10]. In addition to generality, the algorithm contained in [10] has the advantages of being readily understood and yielding solutions valid throughout the time of interest, i.e. for temporal values which preclude asymptotic expansion of the Laplace transform. These qualities have resulted in wide application of the Mindlin–Goodman method as indicated in Meirovitch[11], Epstein[12] and the bibliography in [12]. The essence of the procedure is to extend the method of separation of variables to the solution of partial differential equations governing continuous systems with time dependent boundary conditions. This is accomplished in a system with one spatial variable by expressing the original dependent variable as the sum of a new dependent variable and a finite series. Each term in this series is the product of a boundary condition and an auxiliary function whose argument is the spatial variable. The only constraint on the auxiliary functions is that they contain a sufficient number of terms to render the transformed boundary conditions stationary. Due to the homogeneity of the resulting boundary conditions, the transformed system of equations is amenable to solution by separation of variables. The desired solution of the original time dependent boundary value problem is then obtained from the stated transformation equation. As inspection of the solution given in [10] for a vibrating beam restrained by time dependent boundary conditions reveals, the spatial dependence of the auxiliary functions required in a given problem must be explicitly stated and the integrals containing these functions must be evaluated to obtain the desired solution.

Reflection on the technique outlined above results in the conclusion that the introduction of auxiliary functions is a mathematical convenience, and that the possibility exists of deriving solutions that do not contain such functions. This follows since well posed problems in partial differential equations are known to have unique solutions, i.e. solutions that are independent of the particular auxiliary functions utilized. For one-dimensional wave propagation in a finite, nonhomogeneous bar with time dependent displacement and/or stress boundary conditions, this paper demonstrates a technique, the Boundary Operator Method (BOM), that eliminates the auxiliary functions from the solutions for the displacement and stress in the bar. The generality of the BOM is identical to [10] and, for this reason, the BOM can also be applied to one-dimensional wave propagation problems involving beams, spheres and cylinders composed of homogeneous or nonhomogeneous material.

## ANALYSIS

Based on [7], the equation of motion for the free longitudinal motion in a thin bar with a lengthwise variation in elastic modulus is:

$$\epsilon^n \frac{\partial^2\Psi}{\partial\epsilon^2} + n\epsilon^{n-1}\frac{\partial\Psi}{\partial\epsilon} = \frac{\partial^2\Psi}{\partial\tau^2} \tag{1}$$

where $\Psi(\epsilon, \tau)$ is the dimensionless displacement from the initial position, $n$ is an arbitrary power that describes the axial modulus variation and $\epsilon$ and $\tau$ are the dimensionless spatial and temporal variables, respectively (see Nomenclature). Generalized time dependent boundary conditions are described by:

$$D_i[\Psi(\epsilon_i,\tau)] = \Psi(\epsilon_i,\tau)\cos\alpha_i + \frac{\partial\Psi}{\partial\epsilon}(\epsilon_i,\tau)\sin\alpha_i = f_i(\tau);\ i = 1,2 \tag{2}$$

where $\epsilon_1$ and $\epsilon_2$ correspond to the left and right boundaries of the bar, respectively, and the $\alpha_i$ are real numbers. The associated initial conditions are:

$$\Psi(\epsilon,0) = \Psi_0(\epsilon), \frac{\partial\Psi}{\partial\tau}(\epsilon,0) = \dot{\Psi}_0(\epsilon). \tag{3}$$

Now define the transformation:

$$\Psi(\epsilon,\tau) = \zeta(\epsilon,\tau) + \sum_{i=1}^{2} g_i(\epsilon)f_i(\tau). \tag{4}$$

With the aid of (4), eqns (1)–(3) are written as:

$$\epsilon^n\frac{\partial^2\zeta}{\partial\epsilon^2} + n\epsilon^{n-1}\frac{\partial\zeta}{\partial\epsilon} = \frac{\partial^2\zeta}{\partial\tau^2} - \sum_{i=1}^{2}\{(\epsilon^n g_i'' + n\epsilon^{n-1}g_i')f_i - g_i\ddot{f}_i\} \tag{5}$$

$$D_i[\zeta(\epsilon_i,\tau)] = f_i(\tau) - \sum_{j=1}^{2} D_i[g_j(\epsilon_i)]f_j(\tau);\ i = 1,2 \tag{6}$$

$$\begin{aligned}\zeta(\epsilon,0) &= \Psi_0(\epsilon) - \sum_{i=1}^{2} g_i(\epsilon)f_i(0)\\ \frac{\partial\zeta}{\partial\tau}(\epsilon,0) &= \dot{\Psi}_0(\epsilon) - \sum_{i=1}^{2} g_i(\epsilon)\dot{f}_i(0).\end{aligned} \tag{7}$$

Equation (6) is made homogeneous by requiring that:

$$D_i[g_j(\epsilon_i)] = \delta_{ij},\quad i = 1,2; j = 1,2 \tag{8}$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{1j} = 1$ for $i = j$.

A solution to (5) is now sought in the form:

$$\zeta(\epsilon,\tau) = \sum_{k=1}^{\infty} X_k(\epsilon)T_k(\tau) \tag{9}$$

where the $X_k(\epsilon_i)$ satisfy

$$\epsilon^n X_k'' + n\epsilon^{n-1}X_k' = -\lambda_k^2 X_k,\qquad D_i[X_k(\epsilon_i)] = 0 \tag{10}$$

and the rigid body eigenfunction, $X_0$, has been omitted from the summation. Substituting (9) into (5), multiplying both sides by $X_k$, integrating from $\epsilon_1$ to $\epsilon_2$ and utilizing the orthogonality of the eigenfunctions yields:

$$\ddot{T}_k + \lambda_k^2 T_k = I_k^{-1}\sum_{i=1}^{2}\int_{\epsilon_1}^{\epsilon_2}[(\epsilon^{*n}g_i'' + n\epsilon^{*n-1}g_i')f_i - g_i\ddot{f}_i]X_k\,\mathrm{d}\epsilon^* \tag{11}$$

where $I_k = \int_{\epsilon_1}^{\epsilon_2} X_k^2\,\mathrm{d}\epsilon^*$.

Now employing (8) and (10), the equations:

$$g_j(\epsilon_i)X_k'(\epsilon_i) - g_j'(\epsilon_i)X_k(\epsilon_i) = [-X_k(\epsilon_i)\sin\alpha_i + X_k'(\epsilon_i)\cos\alpha_i]\delta_{ij},\ i = 1,2; j = 1,2 \tag{12}$$

are derived. Thus integrating the first term in the numerator of (11) by parts, invoking (12) and defining the boundary operator:

$$\begin{aligned} 0_i[X_k(\epsilon_i)] &= (-1)^i(\epsilon_i)^n[X_k(\epsilon_i)\sin\alpha_i - X_k'(\epsilon_i)\cos\alpha_i],\ i=1,2;\ \epsilon_i\neq 0 \\ 0_i[X_k(\epsilon_i)] &= (-1)^i[X_k(\epsilon_i)\sin\alpha_i - X_k'(\epsilon_i)\cos\alpha_i],\ i=1,2;\ n=0 \ \text{ and one of } \epsilon_i=0 \end{aligned} \tag{13}$$

simplifies (11) to:

$$\ddot{T}_k + \lambda_k^2 T_k = I_k^{-1}\sum_{i=1}^{2}\left\langle 0_i[X_k(\epsilon_i)]f_i - (\lambda_k^2 f_i + \ddot{f}_i)\int_{\epsilon_1}^{\epsilon_2} g_i X_k\, \mathrm{d}\epsilon^*\right\rangle. \tag{14}$$

The solution of (14) is:

$$T_k(\tau) = A_k\cos\lambda_k\tau + B_k\sin\lambda_k\tau + (\lambda_k I_k)^{-1}\sum_{i=1}^{2}\int_0^{\tau}\left\langle 0_i[X_k(\epsilon_i)]f_i - (\lambda_k^2 f_i + \ddot{f}_i)\int_{\epsilon_1}^{\epsilon_2} g_i X_k\,\mathrm{d}\epsilon^*\right\rangle \times\sin\lambda_k(\tau-\tau^*)\,\mathrm{d}\tau^*. \tag{15}$$

Integrating the last term in the numerator of (15) by parts permits (15) to be written as:

$$\begin{aligned} T_k(\tau) = A_k\cos\lambda_k\tau + B_k\sin\lambda_k\tau + (\lambda_k I_k)^{-1}\sum_{i=1}^{2}\Big\langle 0_i[X_k(\epsilon_i)]\int_0^{\tau} f_i(\tau^*)\sin\lambda_k(\tau-\tau^*)\,\mathrm{d}\tau^* \\ + [\dot{f}_i(0)\sin\lambda_k\tau + \lambda_k f_i(0)\cos\lambda_k\tau - \lambda_k f_i(\tau)]\int_{\epsilon_1}^{\epsilon_2} g_i X_k\,\mathrm{d}\epsilon^*\Big\rangle. \end{aligned} \tag{16}$$

Hence,

$$\begin{aligned} \dot{T}_k(\tau) = -\lambda_k A_k\sin\lambda_k\tau + \lambda_k B_k\cos\lambda_k\tau + I_k^{-1}\sum_{i=1}^{2}\Big\langle 0_i[X_k(\epsilon_i)]\int_0^{\tau} f_i(\tau^*)\cos\lambda_k(\tau-\tau^*)\,\mathrm{d}\tau^* \\ + [\dot{f}_i(0)\cos\lambda_k\tau - \lambda_k f_i(0)\sin\lambda_k\tau - \dot{f}_i(\tau)]\int_{\epsilon_1}^{\epsilon_2} g_i X_k\,\mathrm{d}\epsilon^*\Big\rangle. \end{aligned} \tag{17}$$

From (7) and (9)

$$\begin{aligned} T_k(0) &= I_k^{-1}\int_{\epsilon_1}^{\epsilon_2}[\Psi_0(\epsilon^*) - \sum_{i=1}^{2} g_i(\epsilon^*)f_i(0)]X_k\,\mathrm{d}\epsilon^* \\ \dot{T}_k(0) &= I_k^{-1}\int_{\epsilon_1}^{\epsilon_2}[\dot{\Psi}_0(\epsilon^*) - \sum_{i=1}^{2} g_i(\epsilon^*)\dot{f}_i(0)]X_k\,\mathrm{d}\epsilon^*. \end{aligned} \tag{18}$$

Specifying $\tau = 0$ in (16) and (17) and comparing with (18) yields:

$$\begin{aligned} A_k &= I_k^{-1}\int_{\epsilon_1}^{\epsilon_2}[\Psi_0(\epsilon^*) - \sum_{i=1}^{2} g_i(\epsilon^*)f_i(0)]X_k\,\mathrm{d}\epsilon^* \\ B_k &= (\lambda_k I_k)^{-1}\int_{\epsilon_1}^{\epsilon_2}[\dot{\Psi}_0(\epsilon^*) - \sum_{i=1}^{2} g_i(\epsilon^*)\dot{f}_i(0)]X_k\,\mathrm{d}\epsilon^* \end{aligned} \tag{19}$$

$$\begin{aligned} T_k(\tau) = I_k^{-1}\Big\langle\int_{\epsilon_1}^{\epsilon_2}\Big\{\Psi_0(\epsilon^*)\cos\lambda_k\dot{\tau} + \lambda_k^{-1}\dot{\Psi}_0(\epsilon^*)\sin\lambda_k\tau - \sum_{i=1}^{2} g_i(\epsilon^*)f_i(\tau)\Big\}X_k\,\mathrm{d}\epsilon^* \\ + \lambda_k^{-1}\sum_{i=1}^{2} 0_i[X_k(\epsilon_i)]\int_0^{\tau} f_i(\tau^*)\sin\lambda_k(\tau-\tau^*)\,\mathrm{d}\tau^*\Big\rangle. \end{aligned} \tag{20}$$

Substituting (20) into (9) and employing the result in (4) gives:

$$\begin{aligned} \Psi(\epsilon,\tau) = \sum_{k=1}^{\infty} I_k^{-1}\Big\langle\int_{\epsilon_1}^{\epsilon_2}\Big\{\Psi_0(\epsilon^*)\cos\lambda_k\tau + \lambda_k^{-1}\dot{\Psi}_0(\epsilon^*)\sin\lambda_k\tau - \sum_{i=1}^{2} g_i(\epsilon^*)f_i(\tau)\Big\}X_k\,\mathrm{d}\epsilon^* \\ + \lambda_k^{-1}\sum_{i=1}^{2} 0_i[X_k(\epsilon_i)]\int_0^{\tau} f_i(\tau^*)\sin\lambda_k(\tau-\tau^*)\,\mathrm{d}\tau^*\Big\rangle X_k(\epsilon) + \sum_{i=1}^{2} g_i(\epsilon)f_i(\tau). \end{aligned} \tag{21}$$

Noting the eigenfunction expansion:

$$g_i(\epsilon)=\sum_{k=1}^{\infty} I_k^{-1}\left[\int_{\epsilon_1}^{\epsilon_2} g_i(\epsilon^*)X_k\,\mathrm{d}\epsilon^*\right]X_k(\epsilon),\ \epsilon_1<\epsilon<\epsilon_2 \tag{22}$$

(21) becomes:

$$\Psi(\epsilon,\tau)=\sum_{k=1}^{\infty} I_k^{-1}\left\langle\int_{\epsilon_1}^{\epsilon_2}\{\Psi_0(\epsilon^*)\cos\lambda_k\tau+\lambda_k^{-1}\dot{\Psi}_0(\epsilon^*)\sin\lambda_k\tau\}X_k\,\mathrm{d}\epsilon^*\right.$$
$$\left.+\lambda_k^{-1}\sum_{i=1}^{2}0_i[X_k(\epsilon_i)]\int_0^{\tau} f_i(\tau^*)\sin\lambda_k(\tau-\tau^*)\,\mathrm{d}\tau^*\right\rangle X_k(\epsilon) \tag{23}$$
for $\epsilon_1<\epsilon<\epsilon_2$.

For zero initial conditions and a forcing function applied at the left boundary, (23) reduces to (20) of [7].

Term by term differentation of (23) yields:

$$\frac{\partial\Psi}{\partial\epsilon}(\epsilon,\tau)=\sum_{k=1}^{\infty} I_k^{-1}\left\langle\int_{\epsilon_1}^{\epsilon_2}\{\Psi_0(\epsilon^*)\cos\lambda_k\tau+\lambda_k^{-1}\dot{\Psi}_0(\epsilon^*)\sin\lambda_k\tau\}X_k\,\mathrm{d}\epsilon^*\right.$$
$$\left.+\lambda_k^{-1}\sum_{i=1}^{2}0_i[X_k(\epsilon_i)]\int_0^{\tau} f_i(\tau^*)\sin\lambda_k(\tau-\tau^*)\,\mathrm{d}\tau^*\right\rangle X_k'(\epsilon) \tag{24}$$
for $\epsilon_1<\epsilon<\epsilon_2$.

## NUMERICAL EXAMPLE

The uniaxial stress is evaluated in a homogeneous bar for which the boundary conditions are:

$$f_1(\tau)=\frac{\partial\Psi}{\partial\epsilon}(0,\tau)=\sigma_0\sin\left(\frac{\pi\tau}{\tau_0}\right)\ \text{for } 0\le\tau\le\tau_0$$
$$=0\ \text{for } \tau>\tau_0$$
$$f_2(\tau)=\frac{\partial\Psi}{\partial\epsilon}(1,\tau)=0 \tag{25}$$

Note that (25) implies that $\alpha_1=\alpha_2=\pi/2$ in (2). Thus the eigenfunctions in (9) satisfy free-free boundary conditions. From [7], the required eigenfunctions and eigenvalues for the homogeneous bar are $\cos(k\pi\epsilon)$ and $k\pi$, respectively. The initial conditions are taken as:

$$\Psi_0(\epsilon)=\dot{\Psi}_0(\epsilon)=0. \tag{26}$$

From (23):

$$\Psi(\epsilon,\tau)=-2\sum_{k=1}^{\infty} C_k(\tau)\frac{\cos(k\pi\epsilon)}{k\pi},\qquad 0<\epsilon<1 \tag{27}$$

where:

$$C_k(\tau)=\int_0^{\tau} f_1(\tau^*)\sin k\pi(\tau-\tau^*)\,\mathrm{d}\tau^*. \tag{28}$$

The dimensionless stress is obtained from (24) as:

$$\frac{\partial\Psi}{\partial\epsilon}(\epsilon,\tau)=2\sum_{k=1}^{\infty} C_k(\tau)\sin(k\pi\epsilon),\ 0<\epsilon<1 \tag{29}$$

With $\beta_1 = \pi/\tau_0$ and $\beta_2 = k\pi$, the $C_k(\tau)$ are given as follows for $\tau \le \tau_0$

$$C_k(\tau) = \frac{\sigma_0}{2}\left\langle\left\{\frac{1-\cos(\beta_1-\beta_2)\tau}{\beta_1-\beta_2} + \frac{1-\cos(\beta_1+\beta_2)\tau}{\beta_1+\beta_2}\right\}\sin\beta_2\tau - \left\{\frac{\sin(\beta_1-\beta_2)\tau}{\beta_1-\beta_2} - \frac{\sin(\beta_1+\beta_2)\tau}{\beta_1+\beta_2}\right\}\cos\beta_2\tau\right\rangle, \beta_1 \ne \beta_2 \quad (30)$$

$$= \frac{\sigma_0}{2}\left\langle\frac{\cos\beta_1\tau\sin 2\beta_1\tau - \sin\beta_1\tau\cos 2\beta_1\tau + \sin\beta_1\tau}{2\beta_1} - \tau\cos\beta_1\tau\right\rangle \quad \beta_1 = \beta_2 \quad (31)$$

For $\tau_0 \le \tau$

$$C_k(\tau) = \frac{\sigma_0}{2}\left\langle\left\{\frac{1+\cos\beta_2\tau_0}{\beta_1-\beta_2} + \frac{1+\cos\beta_2\tau_0}{\beta_1+\beta_2}\right\}\sin\beta_2\tau - \left\{\frac{\sin\beta_2\tau_0}{\beta_1-\beta_2} + \frac{\sin\beta_2\tau_0}{\beta_1+\beta_2}\right\}\cos\beta_2\tau\right\rangle, \quad \beta_1 \ne \beta_2 \quad (32)$$

$$= \frac{\sigma_0}{2}\left\langle\frac{\cos\beta_1\tau\sin 2\beta_1\tau_0 - \sin\beta_1\tau\cos 2\beta_1\tau_0 + \sin\beta_1\tau}{2\beta_1} - \tau_0\cos\beta_1\tau\right\rangle, \quad \beta_1 = \beta_2. \quad (33)$$

The behavior of (29) as $\epsilon \to 0$ has been explained by Friedman[13]. For an infinite number of terms in (29), this reference demonstrates that:

$$\frac{\partial\Psi}{\partial\epsilon}(\epsilon, \tau)\Big|_{\epsilon\to 0} = \pi c_1(\tau) \quad (34)$$

where $c_1(\tau)$ is determined from the expansion:

$$C_k(\tau) = \sum_{m=1}^{\infty} \frac{c_m(\tau)}{k^m}. \quad (35)$$

Integration of (28) by parts results in $c_1(\tau) = f_1(\tau)/\pi$. Thus from (34):

$$\frac{\partial\Psi}{\partial\epsilon}\Big|_{\epsilon\to 0} \to \sigma_0 \sin\left(\frac{\pi\tau}{\tau_0}\right), \quad \tau \le \tau_0. \quad (36)$$

For a compressive pulse with amplitude $\sigma_0 = -1.00$ and a time duration $\tau_0 = 0.50$, the digital computer program WAVE, available from the authors upon request, was employed to compute various displacement and stress profiles. Equation (27) was utilized together with the rigid body displacement, $-\int_0^\tau (\sigma_0(\tau-\tau^*)\sin(\pi\tau^*/\tau_0)\,d\tau^*$ from Appendix 2 of [7], to ascertain the axial variation of bar displacement at $\tau = 0.25$, 0.50, 0.75 and 1.00, respectively. The results are shown in Fig. 1. Figures 2 and 3 present curves calculated from (29) for $\tau = 0.25$ and $K_s = 10{,}100$ and 1000, respectively. At this time the amplitude of the pulse at the origin is unity and the leading edge of the pulse has arrived at the location $\epsilon = 0.25$ $[tc_0 = (L\tau/c_0)(c_0) = L\tau,\ \epsilon = L\tau/L = \tau]$. Clearly inclusion of more terms in (29) improves the simulation of the actual stress condition by decreasing the size of the region where the pulse rises from a spurious value of zero to the required unit value of $f_1(0.25)$ and by eliminating the extraneous oscillations near the leading edge of the wave, Fig. 2 also demonstrates that the marked improvement in rise time of the plotted solution for $K_s = 1000$ compared to the curve for $K_s = 100$ is associated with negligible increase in overshooting since the first stress peaks are identical to plotting accuracy. Figure 4 is a plot of two stress profiles in the time regime after pulse application. Since reflection from the free end changes the sign of the incoming stress wave, the spatial variations of the stress at $\tau = 1.5$ and $\tau = 2.0$ are equal to the negative of the solutions obtained for $\tau = 1.0$ and 0.5, respectively.

## SUMMARY

Review of the above numerical example discloses that in order to be assured of the validity of the solution of interest, either in other second order systems or in higher order systems such as govern beams and plates, two conditions must be met. These are that the uniform convergence of any series to be differentiated must be proven (this was done for (24) in [14] and that comparable behavior to that of (36) must be shown at the boundaries of the region which are subjected to time-dependent excitation. In problems where these requirements are satisfied, the analyst gains three essential benefits, in addition to the relative simplicity of the mathematics, from employing the BOM. First, the solutions for the dependent variables of interest are not restricted to a limited range of the temporal variable or a particular set of boundary conditions. In the problem investigated above, the effect of Dirichlet (fixed), Neumann (free) or Cauchy (end constrained by a light spring[15]), boundary conditions can be ascertained by setting $\alpha_1 = \alpha_2 = 0$, $\alpha_1 = \alpha_2 = \pi/2$, $\alpha_1$, $\alpha_2 \neq 0 \neq \pi/2$, respectively in (2). Second,

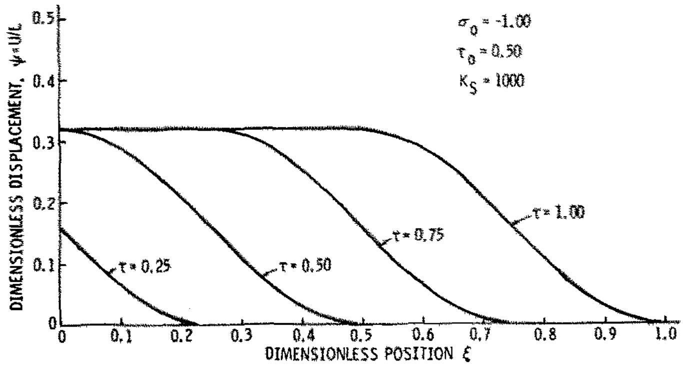


Fig. 1. Displacement solutions during initial wave transversal of bar.

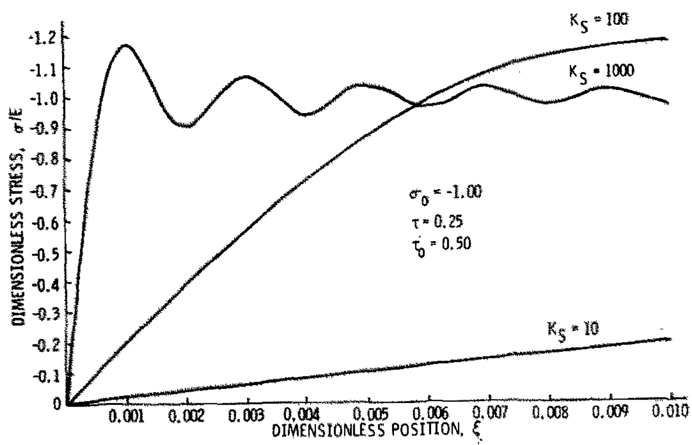


Fig. 2. Stress solution near ($0 < \epsilon \leq 0.01$) forced boundary.

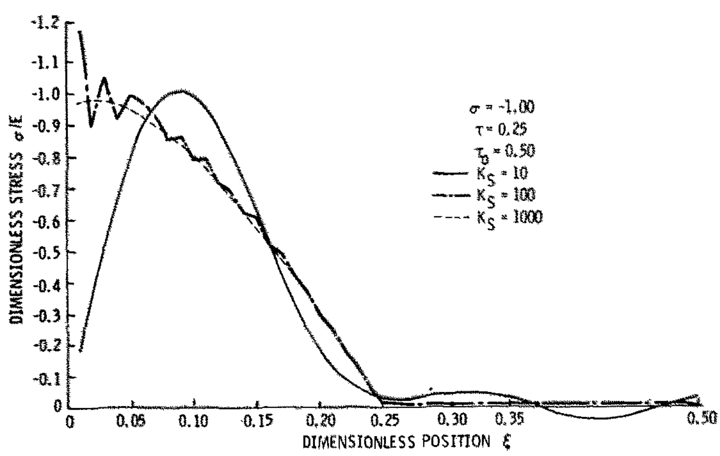


Fig. 3. Stress solution in vicinity ($0.01 \leq \epsilon \leq 0.50$) of forced boundary.

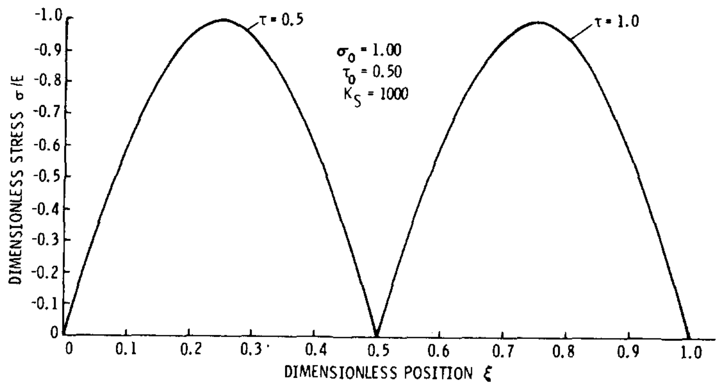


Fig. 4. Stress solution after pulse application.

since auxiliary functions are not employed, the uniqueness of the solution is insured. Third, the necessity of determining and evaluating particular auxiliary functions corresponding to designated boundary conditions which is a requirement of [10] has been eliminated.

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